Journal of Pure and Applied Algebra 72 (1991) 39–51
North-Holland

Newton polygons of jacobian pairs

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Communicated by C.A. Weibel
Received 1 August 1989
Revised 22 November 1989

Abstract

Lang, J., Newton polygons of jacobian pairs, Journal of Pure and Applied Algebra 72 (1991) 39–51.

A pair of elements f, $g \in \mathbb{C}[x, y]$ is a jacobian pair if $(\partial f/\partial x)(\partial g/\partial y) - (\partial f/\partial y)(\partial g/\partial x)$ is a nonzero element of $\mathbb{C}$. The two-variable case of the Jacobian problem asks if every jacobian pair f, g is an automorphic pair. Abhyankar shows in his 1977 Tata notes that an affirmative answer is equivalent to showing that the Newton polygons of a jacobian pair are triangles. In this paper we prove that the Newton polygons of a jacobian pair are similar (a result first presented by Abhyankar in the early 1970's); and that an affirmative answer to the Jacobian problem is equivalent to eliminating the possibility of having edges of positive slope in the Newton polygons of jacobian pairs. In this connection, we investigate ω-related pairs f, $g \in \mathbb{C}[x, y]$ when $\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2$ with $\omega_1\omega_2 < 0$.

Introduction

This article further relates results from Abhyankar's 1977 Tata notes to the study of the Newton polygons of jacobian pairs in $k[x, y]$, k a field of characteristic 0. A pair of elements f, $g \in k[x, y]$ will be referred to as a *jacobian pair* if

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

is a nonzero element of k. An *automorphic* pair f, $g \in k[x, y]$ is one such that $k[f, g] = k[x, y]$. The Jacobian problem asks if every jacobian pair is an automorphic pair in $k[x, y]$. The converse is well known to be true [1, p. 118]. In [1], Abhyankar shows that every jacobian pair is an automorphic pair if and only if the Newton polygons of every jacobian pair are triangles with vertices on the coordinate axes (see Theorem 1.9). Further consequences of Abhyankar's results relating to the Newton polygons of jacobian pairs are derived in this paper using

* Partially supported by the General Research Fund at the University of Kansas.

methods that are, not surprisingly, similar to his [1, pp. 117–145]. McKay and Wang [4, p. 116] prove that Newton polygons of an automorphic pair are similar triangles. In this article we show that the two Newton polygons of a jacobian pair are similar polygons (Theorem 3.4). This result is widely accepted. Abhyankar presented a proof in the early 1970's which is contained in unpublished notes [6]. Also Nowicki and Nakai come very close to this result in [5]. We also show that every jacobian pair is an automorphic pair if and only if the jacobian condition ($f_x g_y - f_y g_x$ is a nonzero element of k) implies that $N(f)$ has no edge of positive slope (Proposition 2.5). This leads in Section 4 to the study of ω-related pairs f, $g \in k[x, y]$ where $\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2$ with $\omega_1 \omega_2 < 0$. Proposition 4.2 is analogous to a result of Abhyankar when $\omega_1 \omega_2 > 0$ (see [1, p. 132]), but it's possible implications need further study (Remark 4.4). Section 5 ties up some loose ends.

1. Preliminaries

We let $\mathbb{Z}$ denote the integers, $\mathbb{Z}^+$ the nonnegative integers, $\mathbb{N}$ the natural numbers, $\mathbb{Q}$ the rationals, $\mathbb{R}$ the reals and $\mathbb{C}$ the complex numbers.

Let k be a field. Given $f = \sum a_{ij} x^i y^j \in k[x, y]$ $(\alpha_{ij} \in k)$, $S(f) = \{(i, j)\colon \alpha_{ij} \neq 0\} \cup \{(0, 0)\}$ and $N(f)$ is the smallest convex subset of $\mathbb{R}^2$ containing $S(f)$. $N(f)$ is called the *Newton polygon* of f.

Let $\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2$. By the ω-gradation on $k[x, y]$ we mean the gradation obtained by assigning weights ω_1 to x and ω_2 to y. For $f \in k[x, y]$, $\deg_\omega f$ denotes the ω-degree of f and f_ω^+ denotes the highest ω-degree form. When $\omega = (1, 1)$ we will simply write f^+ for f_ω^+ and $\deg f$ for $\deg_\omega f$.

Given f, $g \in k[x, y]$,

$$J(f, g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} .$$

$\ominus$ denotes a generic (i.e., unspecified) element of k.

∇ denotes the differential operator

$$\nabla = \frac{\partial^{2p-2}}{\partial x^{p-1} \partial y^{p-1}} .$$

k^* denotes the set of nonzero elements of k.

The following results, 1.1–1.9, are proved in Abhyankar's 1977 Tata notes. k denotes a field of characteristic 0 and $\omega = (\omega_1, \omega_2)$ a pair of integers. $A = k[x, y]$.

Lemma 1.1 [1, p. 120]. *Let F, G be nonzero ω-homogeneous elements of A. The following two conditions are equivalent*:

(1) $F^r = \ominus\, G^s$ *for some* $r, s \in \mathbb{Z}^+$, $r + s > 0$.

(2) *There exist* $p, q \in \mathbb{Z}^+$ *and an ω-homogeneous element H of A such that* $F = \ominus\, H^p$, $G = \ominus\, H^q$. $\square$

Definition 1.2. Let $f, g \in A$. We say that f and g are *ω-related* if $f \neq 0$, $g \neq 0$, and $F = f^+_\omega$ and $G = g^+_\omega$ satisfy the equivalent conditions (1) and (2) of Lemma 1.1.

Proposition 1.3 [1, p. 122]. *Let F, G be nonzero ω-homogeneous elements of A of ω-degrees m, n, respectively. Assume that either* (i) *$\omega_1\omega_2 > 0$ and $FG \not\in k$ or* (ii) *$m \neq 0$ or $n \neq 0$. Then the following statements are equivalent:*

(1) *F and G are ω-related,*
(2) *F and G are algebraically dependent over k,*
(3) $J(F, G) = 0$,
(4) $F^n = \ominus G^m$,
(5) $F^{|n|} = \ominus G^{|m|}$. □

Lemma 1.4 [1, p. 131]. *Let $f \in A$ with $d_\omega(f) \neq 0$. Suppose there exists $g \in A$ such that f and $J(f, g)$ are ω-related. Then there exists ω-homogeneous elements H, G of A, a positive integer p and a nonnegative integer r such that $f^+_\omega = \ominus H^p$ and $J(H, G) = \ominus H^r$.* □

Lemma 1.5 [1, p. 132]. *Assume $\omega_1\omega_2 > 0$. Let $f, g \in A$ such that f and $J(F, g)$ are ω-related. Then there exist ω-homogeneous elements H, G of A and a positive integer p such that $f^+_\omega = \ominus H^p$ and $J(H, G) = \ominus H^s$ with $s = 0$ or 1.* □

Theorem 1.6 [1, p. 138]. *Assume $\omega_1 > 0$, $\omega_2 > 0$. Let f, $g \in A$ such that $J(f, g) = \ominus$. Then $f^+_\omega = \ominus u_1^{i_1} u_2^{i_2}$ with i_1, i_2 nonnegative integers, $i_1 + i_2 > 0$, and $u = (u_1, u_2)$ has one of the following three forms:*

(i) *if $\omega_1 = \omega_2$, then u_i is homogeneous linear in x, y, $i = 1, 2$;*
(ii) *if $\omega_1 > \omega_2$, then $u_1 = x + ay^{\omega_1/\omega_2}$, $u_2 = y$, with $a \in k$ and $\omega_2/\omega_1 \in \mathbb{N}$ if $a \neq 0$;*
(iii) *if $\omega_1 < \omega_2$, then $u_1 = x$, $u_2 = y + ax^{\omega_2/\omega_1}$ with $a \in k$ and $\omega_2/\omega_1 \in \mathbb{N}$ if $a \neq 0$.* □

Definition 1.7. Let $\bar{k}$ be the algebraic closure of k. Let $f \in A$ such that $f \not\in k$ and let $r \in \mathbb{N}$. We say f has *r points at infinity with respect to the ω-gradation* if f^+_ω is a product of r-mutually coprime factors in $\bar{k}[x, y]$. We say f has *r points at infinity* if it has r points at infinity with respect to the $(1, 1)$-gradation.

Corollary 1.8 [1, p. 139]. *Lef $f, g \in A$ such that $J(f, g) = \ominus$. If $\omega_1 > 0$, $\omega_2 > 0$, then f (also g) has at most two points at infinite with respect to the ω-gradation.* □

Theorem 1.9 [1, p. 143]. *The following statements are equivalent:*

(i) *If $f, g \in A$ and $J(f, g) = \ominus$, then $k[f, g] = A$.*
(ii) *If $f, g \in A$ and $J(f, g) = \ominus$, then f has one point at infinity.*
(iii) *If $f, g \in A$ and $J(f, g) = \ominus$, then $N(f)$ is a triangle with vertices $(0, n)$, $(m, 0)$, $(0, 0)$ for some $n, m \in \mathbb{Z}^+$.*
(iv) *If $f, g \in A$ and $J(f, g) = \ominus$, then* $\deg f$ *divides* $\deg g$ *or* $\deg g$ *divides* $\deg f$. □

Theorem 1.10 (Lang [2, p. 395]). *Let k be a field of characteristic p and $A = k[x, y]$. Let $f \in A$ and D be the derivation on $k(x, y)$ defined by $D(h) = J(h, f)$, for all $h \in k(x, y)$. Then $D^p = aD$, where $a = \sum_{i=0}^{p-1} f^{p-i-1}\nabla(f^i)$, $\nabla = \partial^{2p-2}/\partial x^{p-1}\,\partial y^{p-1}$.* $\square$

2. Newton polygons

Definition 2.0. In this section, let k be a field of characteristic 0, $f, g \in k[x, y]$. Let $m = \max\{j: (0, j) \in S(f)\}$, $m' = \max\{j: (0, j) \in S(g)\}$, $n = \max\{i: (i, 0) \in S(f)\}$ and $n' = \max\{i: (i, 0) \in S(g)\}$. Assume $\deg f \geq 2$, $\deg g \geq 2$. Assume $N(f)$ has s vertices and $N(g)$ has t vertices. Proceeding in the clockwise direction from $(0, 0)$ denote them by $A_0 = (0, 0)$, $A_1 = (a_1, b_1), \ldots, A_s = (a_s, b_s)$ and $A'_0 = (0, 0)$, $A'_1 = (a'_1, b'_1), \ldots, A'_t = (a'_t, b'_t)$, respectively. Also let $A_{s+1} = (0, 0)$, $A'_{t+1} = (0, 0)$.

For each $0 \leq i \leq s$, $0 \leq j \leq t$, let $\vec{v}_i$ denote the vector $\vec{v}_i = \overrightarrow{A_iA_{i+1}}$ and $\vec{v}'_j$ denote the vector $\vec{v}'_j = \overrightarrow{A'_jA'_{j+1}}$ (see Fig. 1).

Lemma 2.1. *With f as in Definition 2.0, let $\omega_1, \omega_2 \in \mathbb{Z}$ with $\omega_1 + \omega_2 \neq 0$. If the angle between $\vec{\omega} = \omega_1\vec{i} + \omega_2\vec{j}$ and $\vec{v}_k$ is $\pi/2$ measured in the clockwise direction from $\vec{\omega}$, then f^+_ω has the form $\Theta_1 x^{a_k}y^{b_k} + \cdots + \Theta_2 x^{a_{k+1}}y^{b_{k+1}}$ where $\Theta_1, \Theta_2 \in k^*$. Furthermore, if the angle between $\vec{\omega}$ and $\vec{v}_k$ is not $\pi/2$ for each $k = 0, 1, \ldots, s$, then $f^+_\omega = \Theta x^{a_r}y^{b_r}$ for some r.*

Proof. Let (x', y) be the coordinate system on $\mathbb{R}^2$ having positive y' direction in the direction of $\vec{\omega}$ and origin at $A_0 = (0, 0)$ (see Fig. 2).

Let A be a point of $N(f)$ with x coordinate a and y coordinate b. Let $\vec{u} = \overrightarrow{A_0A}$. Then with respect to the x'–y' coordinate system the y' coordinate of A is given

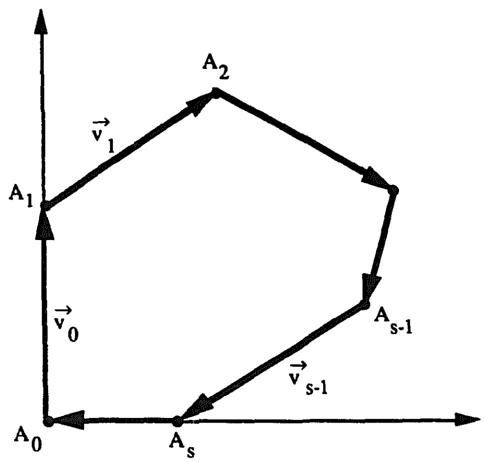

Fig. 1.

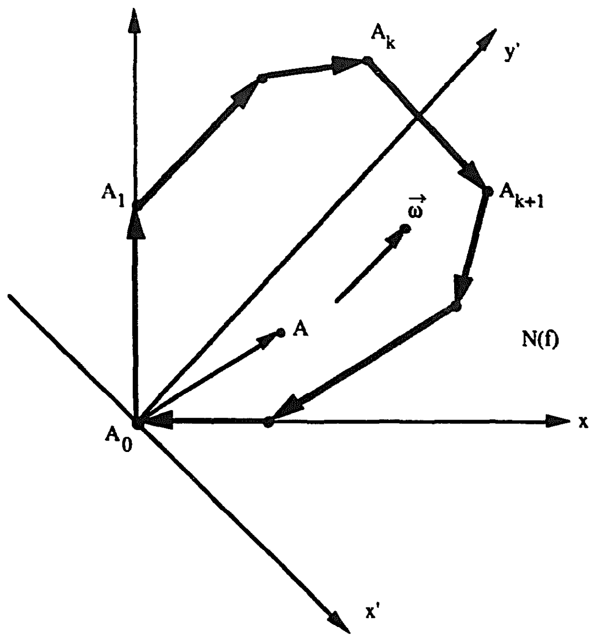

Fig. 2.

by

$$\frac{\vec{n}\cdot\vec{\omega}}{\|\vec{\omega}\|} = (a\omega_1 + b\omega_2)(\omega_1^2 + \omega_2^2)^{-1/2} . \tag{1}$$

If the angle between $\vec{\omega}$ and $\vec{v}_k$ is $\pi/2$, then all points of $S(f) \cap \overline{A_k A_{k+1}}$ have the same y' coordinate and all other points of $S(f)$ have lesser y' coordinate. This implies that $f_\omega^+ = \ominus_1 x^{a_k} y^{b_k} + \cdots + \ominus_2 x^{a_{k+1}} y^{b_{k+1}}$ where $\ominus_1, \ominus_2 \in k^*$.

If the angle between $\vec{\omega}$ and $\vec{v}_k$ is not $\pi/2$ for each $1 \le k \le s$, then there exists a unique $r = 1, \ldots, s$ such that the y' coordinate of A_r is strictly greater than the y' coordinate of each point of $N(f)$. By (1) this shows that $f_\omega^+ = \ominus x^{a_r} y^{b_r}$. $\square$

Lemma 2.2. *With f, g as in Definition* 2.0, *if $J(f, g) = \ominus$, then $nm \neq 0$.*

Proof. For all $h \in A$ and $i \in \mathbb{Z}^+$, denote by h_i the homogenous form of degree i of h. We may assume $f_0 = g_0 = 0$. $J(f, g) = \ominus$ implies $J(f_1, g_1) = \ominus$. Thus $f = \alpha_1 x + a_2 y$, $g_1 = \beta_1 x + \beta_2 y$ with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in k$ and $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. We may therefore assume that $\alpha_1 \neq 0$. If $\alpha_2 \neq 0$, we are done. If $\alpha_2 = 0$, then $\beta_2 \neq 0$ and hence $m' \neq 0$. Thus $A_1' = (0, m')$. Recall $A_2' = (a_2', b_2')$. If $a_2' = 0$, then $A_2' = A_0'$. This would imply that $g \in k[y]$ and $\deg(g) = 1$. Therefore $a_2' \neq 0$.

Let $\omega_1 = m' - b_2'$, $\omega_2 = a_2'$ and $\omega = (\omega_1, \omega_2)$. By Lemma 2.1, g_ω^+ has the form $g_\omega^+ = \ominus_1' x^{a_2'} y^{b_2'} + \cdots + \ominus_2' y^{m'}$, with $\ominus_1', \ominus_2' \in k^*$.

We have that $J(f_\omega^+, g_\omega^+)$ equals 0 or $\ominus$. By Proposition 1.3 the first possibility implies there exist positive integers N, M such that $(f_\omega^+)^N = \ominus(g_\omega^+)^M$. It then follows that f_ω^+ has the form $\Theta_1 x^a y^b + \cdots + \Theta_2 y^t$, with $\Theta_1, \Theta_2 \in k^*$, $t \in \mathbb{N}$. Hence $m \neq 0$.

Suppose now that $J(f_\omega^+, g_\omega^+) = \Theta$. Let $\bar{f} = f_\omega^+$, $\bar{g} = g_\omega^+$. Then $\bar{g}_1 \neq 0$. Thus either $m' = 1$ or $a_2' = 1$ and $b_2' = 0$.

Case 1. Assume $a_2' = 1$ and $b_2' = 0$. Then $A_2' = (1, 0)$ and $\bar{g}$ must be of the form $\Theta_1 x + \Theta_2 y^{m'}$ with $m' > 1$, since $\deg(g) \geq 2$. Either $J(\bar{f}^+, \bar{g}^+) = 0$ or $\deg J(\bar{f}^+, g^+) > 0$. Since $J(\bar{f}, \bar{g}) = \Theta$, we obtain $J(\bar{f}^+, \bar{g}^+) = 0$. By Proposition 1.3, $\bar{f}^+ = \Theta y^r$ for some $r \in \mathbb{N}$ and $m' \neq 0$.

Case 2. Assume $m' = 1$.

If $b_2' = 0$, then $A_2' = (a_2', 0)$ and $\bar{g} = \Theta_1' x^{a_2'} + \Theta_2' y$. Since $\deg(g) \geq 2$, $a_2' \geq 2$. Then $J(\bar{f}, \bar{g}) = \Theta$ implies $J(\bar{f}_1, y) = \Theta$. Therefore $\bar{f}_1 = \gamma_1 x + \gamma_2 y$ with $\gamma_1 \in k^*$, $\gamma_2 \in k$. Arguing as above, $J(\bar{f}^+, \bar{g}^+) = 0$. By Proposition 1.3 $\bar{f}^+ = \Theta x^r$ for some $r \in \mathbb{Z}^+$. Since $\gamma_1 \neq 0$ and $\bar{f}$ is ω-homogeneous, $\deg_\omega(x) = \deg_\omega(x^r)$. Hence $r = 1$. This is impossible since $\deg f \geq 2$, $\omega_1 = 1$ and $\omega_2 \geq 2$.

Suppose $b_2' > 0$. We saw above that $a_2' \neq 0$. Thus $\bar{g}^+ = \Theta x^{a_2'} y^{b_2'}$ and $\bar{g}_1 = \Theta y$. It then follows that $\bar{f}_1 = \Theta x$ and by Proposition 1.3 $\bar{f}^+ = \Theta x^a y^b$ with $ab_2' = a_2' b$, $a, b \in \mathbb{N}$. Since $\bar{f}$ and $\bar{g}$ are ω-homogeneous, the line segment through $(0, 1)$ and (a_2', b_2') and the line segment through $(1, 0)$ and (a, b) are parallel. Then $(b_2' - 1)/a_2' = b/(a-1)$, which gives $a + b_2' = 1$. Contradiction. □

Proposition 2.3. *Let f, g, n, m be as in Definition 2.0. Assume $J(f, g) = \Theta$. Suppose for some $k = 1, \ldots, s$, the line segment $\overline{A_k A_{k+1}}$ has negative slope. Then $A_k = (0, m)$ or $A_{k+1} = (n, 0)$.*

Proof. By Lemma 2.2, $nm \neq 0$. Let $\omega_1 = b_k - b_{k-1}$, $\omega_2 = a_{k+1} - a_k$, $\omega = (\omega_1, \omega_2)$. Then $\omega_1 > 0$ and $\omega_2 > 0$ (see Fig. 3).

With respect to the ω-gradation, f_ω^+ has the form $f_\omega^+ = \Theta_1 x^{a_k} y^{b_k} + \cdots + \Theta_2 x^{a_{k+1}} y^{b_{k+1}} = x^{a_k} y^{b_{k+1}} (\Theta_1 y^{\omega_1} + \cdots + \Theta_2 x^{\omega_2})$, where $\Theta_1, \Theta_2 \in k^*$ by Lemma 2.1.

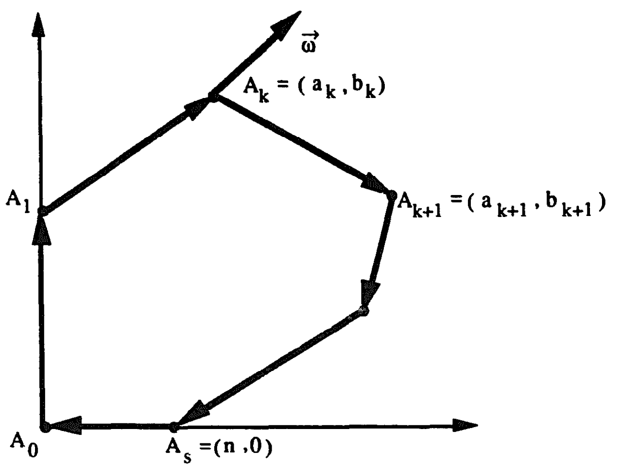

Fig. 3.

If $a_k b_{k+1} \neq 0$, then f has at least three points at infinity with respect to the ω-gradation. Thus by Corollary 1.8, $a_k = 0$ or $b_{k+1} = 0$. If $a_k = 0$, then $A_k = A_1 = (0, m)$ and if $b_{k+1} = 0$, then $A_{k+1} = A_s = (n, 0)$. □

Corollary 2.4. *Let f, g, n, m be as in Definition 2.0. Assume that $J(f, g) = \ominus$ and $N(f)$ is contained in the half-open rectangle $R^h = \{(x, y): 0 \leq x < n, 0 \leq y < m\} \cup \{(n, 0), (0, m)\}$. Then $N(f)$ is a triangle with vertices $(0, 0)$, $(n, 0)$, $(0, m)$ or is a quadrilateral with additional vertex (a, b) with $0 < a < n$, $0 < b < m$.* □

Proposition 2.5. *Let f, g be as in Definition 2.0, R^h as in Corollary 2.4 and R the closed rectangle $R = \{(x, y): 0 \leq x \leq n, 0 \leq y \leq m\}$. Then the following are equivalent:*

(a) *$J(f, g) = \ominus$ implies $k[f, g] = k[x, y]$.*
(b) *$J(f, g) = \ominus$ implies $N(f) \subseteq R^h$.*
(c) *$J(f, g) = \ominus$ implies $N(f) \subseteq R$.*
(d) *$J(f, g) = \ominus$ implies the line segment $\overline{A_1 A_2}$ has slope less than 0.*

Proof. (a)$\Rightarrow$(b) By Theorem 1.9.

(b)$\Rightarrow$(a) By Theorem 1.9 and Corollary 2.4 it is enough to show that (b) implies $N(f)$ is not a quadrilateral if $J(f, g) = \ominus$. Assume it is with fourth vertex (a, b) with $0 < a < n$, $0 < b < m$ (see Fig. 4).

We proceed by induction on $n - a$ to show that this is not possible. If $n - a = 1$, then $a = n - 1$. Let $\omega_1 = b$, $\omega_2 = 1$, $\omega = (\omega_1, \omega_2)$. By Lemma 2.1 f_ω^+ has the form $f_\omega^+ = \ominus_1 x^n + \cdots + \ominus_2 x^a y^b = x^a(\ominus_1 x + \cdots + \ominus_2 y^b)$ with $\ominus_1$, $\ominus_2 \in k^*$. If $\omega_2 < \omega_1$, then by Theorem 1.6, y is a factor of f_ω^+. Thus $1 = \omega_2 = \omega_1 = b$ and $f_\omega^+ = x^{n-1}(\ominus_1 x + \ominus_2 y)$. Make the change of coordinates $x \to x$, $y \to \ominus_2^{-1}(y - \ominus_1 x)$. Then $N(f)$ will still be a quadrilateral with vertices $(0, 0)$, $(0, m)$, (a, b) and $(n', 0)$ with $n' < n$. Since $a = n - 1$, this contradicts (b).

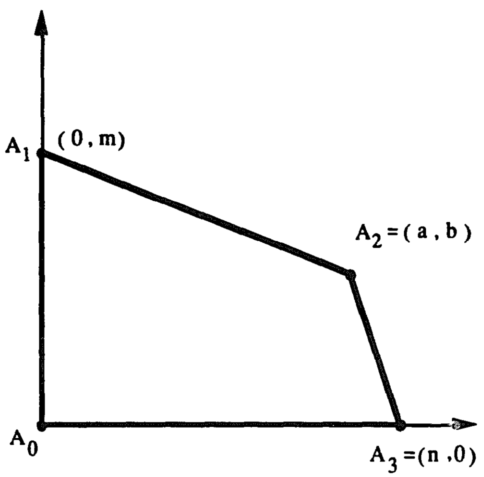

Fig. 4.

Assume $n-a>1$. Let $\omega_1=b$, $\omega_2=n-a$. Then f_ω^+ has the form $f_\omega^+=\ominus_1 x^n+\cdots+\ominus_2 x^a y^b=x^a(\ominus_1 x^{n-a}+\cdots+\ominus_2 y^b)$, $\ominus_1, \ominus_2\in k^*$. By Theorem 1.6 $b^{-1}(n-a)=\omega_1^{-1}\omega_2\in\mathbb{N}$ and $\ominus_1 x^{n-a}+\cdots+\ominus_2 y^b=\ominus(y+ax^{\omega_1^{-1}\omega_2})^s$, $a\in k^*$, $s\in\mathbb{N}$. Make the change in coordinates $x\to x$, $y\to y-ax^{\omega_1^{-1}\omega_2}$. By Corollary 2.4 and (b), $N(f)$ will still be a quadrilateral with vertices $(0,m)$, $(0,0)$, (a,b), and $(n',0)$ with $n'<n$ and $a<n'$. We then have $n'-a<n-a$.

(b)$\Rightarrow$(c) Clear.

(c)$\Rightarrow$(d) Assume $J(f,g)=\ominus$. By (b) we need only show that $\overline{A_1A_2}$ is not horizontal. Suppose that the slope of $\overline{A_1A_2}$ is 0. Then $A_2=(b_2,m)$. Let $\omega_1=(0,1)$. By Lemma 2.1, $f_\omega^+=\ominus_1 y^m+\cdots+\ominus_2 x^{a_2}y^m=y^m(\ominus_1+\cdots+\ominus_2 x^{a_2})$, $\ominus_1, \ominus_2\in k^*$. Choose $\alpha\in k$ such that $\ominus_2\alpha^{a_2}+\cdots+\ominus_1=0$. Make the change in coordinates $x\to x+\alpha$, $y\to y$. Then the jacobian pair we obtain has line segment $\overline{A_1A_2}$ with positive slope, contradicting (c).

(d)$\Rightarrow$(a) Assume (d) and that f, g is a jacobian pair. Then $\overline{A_1A_2}$ has slope less than 0. Let $\tilde{f}=f(y,x)$ and $\tilde{g}=g(y,x)$. Then $J(\tilde{f},\tilde{g})=\ominus$ and $N(\tilde{f})$ is a reflection of $N(f)$ about the line $y=x$. By (d), $\overline{A_{s-1}A_s}$ has a negative slope, hence $N(f)\subseteq R^{\mathrm{h}}$. □

Remark 2.6. Proposition 2.5 suggests the study of f_ω^+ when $\omega_1\omega_2<0$. This we begin in Section 4, primarily for the case when $\overline{A_1A_2}$ has positive slope less than 1. Before doing this, we prove that Newton polygons of jacobian pairs are similar.

3. $J(f,g)=\ominus$ implies $N(f)$ and $N(g)$ are similar

Definition 3.0. In this section, f, g, $\vec{v}_i$ $(0\le i\le s)$, $\vec{v}_j'$ $(0\le j\le t)$ are as in Definition 2.0.

Proposition 3.1. *$J(f,g)=\ominus$ implies that for each $0\le i\le s$, there exists a $\vec{v}_j'$ in the same direction as $\vec{v}_i$.*

Proof. Suppose $\vec{v}_i$ is such that no $\vec{v}_j'$ has the same direction. Let $\omega_i=b_i-b_{i+1}$, $\omega_2=a_{i+1}-a_i$, $\omega=(\omega_1,\omega_2)$ (see Fig. 5).

By Lemma 2.1, f_ω^+ has the form $f_\omega^+=\ominus_1 x^{a_i}y^{b_i}+\cdots+\ominus_2 x^{a_{i+1}}y^{b_{i+1}}$ and g_ω^+ has the form $g_\omega^+=\ominus x^{a_j'}y^{b_j'}$ for some $j=0,1,\ldots,t$, where $\ominus_1\ominus_2\neq 0$. This contradicts Lemma 1.3. □

Corollary 3.2. *$J(f,g)=\ominus$ implies $s=t$ (i.e. $N(f)$ and $N(g)$ have the same number of vertices). Furthermore, the direction of v_i is the same as that of v_i' for each $i=0,\ldots,s$.*

Proof. Let $\vec{u}$ be the unit vector in the direction of $\vec{v}_0$ (equivalently $\vec{v}_0'$). Then for each $0\le i\le s$, the vector direction of $\vec{v}_i$ is determined uniquely by the angle θ_i

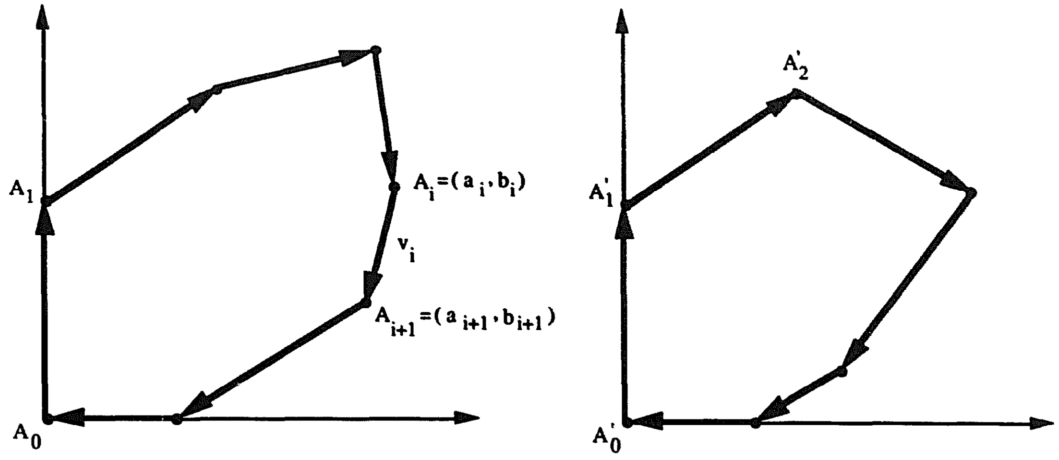

Fig. 5.

between $\vec{u}$ and $\vec{v}_i$ measured in the clockwise direction from $\vec{u}$. Similarly, the angle θ'_j, $0 \le j \le t$, measured in the clockwise direction from $\vec{u}$ to $\vec{v}'_j$ uniquely determines the direction of $\vec{v}'_j$. We have strictly ascending sequences, $0 = \theta_0 < \theta_1 < \cdots < \theta_s = 3\pi/2$ and $0 = \theta'_0 < \theta'_1 < \cdots < \theta'_t = 3\pi/2$. Now apply Proposition 3.1. □

Corollary 3.3 (MacKay and Wang). *Let $f, g \in k[x, y]$. If $k[f, g] = k[x, y]$, then $N(f)$ and $N(g)$ are similar triangles.*

Proof. That $N(f)$ and $N(g)$ are triangles was proved by Abhyankar [1, p. 144]. That they are similar follows from Corollary 3.2. □

Theorem 3.4. *Let f, g be as in Definition 3.0. Then $J(f, g) = \Theta$ implies $N(f)$ and $N(g)$ are similar polygons. That is, for each i, v_i and v'_i have the same direction and for each i, j, $0 \le i, j \le s$, $\|v_i\| / \|v_j\| = \|v'_i\| / \|v'_j\|$.*

Proof. For each $i = 1, \ldots, s$, let $\omega(i) = (b_i - b_{i+1}, a_{i+1} - a_i)$. Then by Lemma 2.1 and Corollary 3.2, $f^+_{\omega(i)}$ and $g^+_{\omega(i)}$ have the forms $f^+_{\omega(i)} = \Theta_{i1} x^{a_i} y^{b_i} = \cdots + \Theta_{i2} x^{a_{i+1}} y^{b_{i+1}}$, $g^+_{\omega(i)} = \Theta'_{i1} x^{a'_i} y^{b'_i} + \cdots + \Theta'_{i2} x^{a'_{i+1}} y^{b'_{i+1}}$, with $\Theta_{i1}\Theta_{i2}\Theta'_{i1}\Theta'_{i2} \neq 0$. By Lemma 1.3, for each i, $0 \le i \le s$, there exists N_i, $M_i \in \mathbb{N}$ such that $(f^+_{\omega(i)})^{N_i} = (g^+_{\omega(i)})^{M_i}$. Comparing monomials on both sides of these equalities we obtain

$$a_i N_i = a'_i M_i \quad \text{and} \quad b_i N_i = b'_i M_i\,, \qquad 0 \le i \le s\,, \tag{2}$$

and

$$a_{i+1} N_i = a'_{i+1} M_i \quad \text{and} \quad b_{i+1} N_i = b'_{i+1} M_i\,, \quad 0 \le i \le s\,. \tag{3}$$

From (2) and (3) we obtain

$$\frac{a_i}{a_i'} = \frac{a_{i+1}}{a_{i+1}'} = \frac{b_i}{b_i'} = \frac{b_{i+1}}{b_{i+1}'}, \quad 0 \le i \le s. \tag{4}$$

Let $r \in \mathbb{Q}$ be this common ratio. Then

$$a_i = ra_i' \quad \text{and} \quad b_i = rb_i', \qquad 0 \le i \le s. \tag{5}$$

Thus $\|\vec{v}_i\| / \|\vec{v}_i'\| = r$, $1 \le i \le s$. $\square$

4. Edges of $N(f)$ with positive slope

Let $F(t), G(t) \in k[t]$ such that $F(0)G(0) \neq 0$. Let $a, b, p, q, p', q' \in \mathbb{Z}^+$ such that $pq' - p'q \neq 0$ and $(bp - aq)(b - a) > 0$. Let $f, g \in k[x, y]$ be given by $f = x^p y^q F(x^a y^b)$, $g = x^{p'} y^{q'} G(x^a y^b)$.

Lemma 4.1. *Suppose $J(f, g) = \ominus f^A$ for some $A \in \mathbb{N}$. Then f^{A-1} divides g.*

Proof.

$$\begin{aligned}\ominus f^A &= J(f, g)\\ &= x^p y^q G J(F, x^{p'} y^{q'}) + x^p y^{q'} F J(y^q, x^{p'} G) + x^{p'} y^q F J(x^p, y^{q'} G)\\ &= x^{p+p'-1} y^{q+q'-1} [x^a y^b ((aq' - bp') F' G + (bp - aq) F G'\\ &\qquad + (pq' - p'q) F G)].\end{aligned}$$

Therefore

$$\begin{aligned}&\ominus x^{p(A-1)-p'+1} y^{q(A-1)-q'+1} F^A\\ &\quad = x^a y^b [(aq' - bp') F G' + (pb - aq) F G'] + (pq' - p'q) F G. \end{aligned} \tag{6}$$

The expression to the right of the equality in (6) belongs to $k[x^a y^b]$, as does F^A. Since $F(0)G(0) \neq 0$ and $pq' - p'q \neq 0$, we have

$$\begin{aligned}&p(A-1) = p' - 1, \qquad q(A-1) = q' - 1,\\ &\ominus (F(t))^A = t[(aq' - bp') F' G - (aq - bp) F G'] + (pq' - p'q) F G.\end{aligned} \tag{7}$$

$p(A-1) = p' - 1$ and $q(A-1) = q' - 1$ implies $qp' - q = pq' - p$, hence $pq' - p'q = p - q$. Therefore

$$\ominus F^A = t[(aq' - bp') F' G - (aq - bp) F G'] + (p - q) F G. \tag{8}$$

Since $pq' - p'q \neq 0$, we may without loss of generality assume that $p \neq 0$. Then

$$\begin{aligned} A - 1 < \frac{aq' - bp'}{aq - bp} &\Leftrightarrow \frac{p' - 1}{p} < \frac{aq' - bp'}{aq - bp} \\ &\Leftrightarrow \frac{(aq - bp)(p' - 1) - (aq' - bp')p}{(aq - bp)p} < 0 \\ &\Leftrightarrow \frac{a(p'q - pq') - aq + bp}{p(aq - bp)} < 0 \\ &\Leftrightarrow \frac{p(b - a)}{p(aq - bp)} < 0 \Leftrightarrow (bp - aq)(b - a) > 0 . \end{aligned}$$

By Lemma 4.2 below, we conclude that F^{A-1} divides G. Since $p(A-1) < p'$ and $q(A-1) < q'$ we have f^{A-1} divides g in $k[x, y]$. $\square$

Lemma 4.2. *Let $F, G \in k[t]$, $F(0)G(0) \neq 0$. Let $A, B, M, N \in \mathbb{Z}$ such that $M \geq 0$, $N > 0$, $A > 0$, $M(A-1) \leq N$ and $NtF'G - MtFG' + BFG = \ominus F^A$. Then F^{A-1} divides G.*

Proof. We may assume that k is algebraically closed. Then F is a product in $k[t]$ of linear polynomials. It is enough to prove that if l is a factor of F and p is a positive integer such that l^p divides F, then $l^{p(A-1)}$ divides G. We have $l = at + b$ with $b \neq 0$ since $F(0) \neq 0$. After a change in coordinates we may assume that $a = 1$. We may also assume that l^{p+1} does not divide F. Write $F = l^p F_0$, $G = l^q G_0$ with $q \in \mathbb{Z}^+$, $f_0 G_0 \neq 0 \pmod l$. We will show $q \geq (A-1)p$.

Case 1: $Np = Mq$. Then $M > 0$ and $q/p = N/M \geq A - 1$. Hence $q \geq (A-1)p$.

Case 2: $Np \neq Mq$. Since $\mathrm{d}l/\mathrm{d}t = 1$, we have $F' = pl^{p-1}F_0 \pmod{l^p}$ and $G' = gl^{q-1}G_0 \pmod{l^q}$. Therefore,

$$NtF'G - MtFG' - BFG = (Np - Mq)l^{p+q-1}tF_0G_0 \pmod{l^{p+q}} .$$

Since $Np - Mq \neq 0$, $F_0G_0 \neq 0 \pmod l$ and since F_A divides $NtF'G - MtF' - BFG$, we get $pA \leq p + q - 1$. Hence $p(A-1) \leq q - 1 < q$. $\square$

Proposition 4.3. *Let f, g, A_1, A_2 be as in Definition* 2.0. *Assume $J(f, g) = \ominus$. Let $\omega_1 = b_1 - b_2$, $\omega_2 = a_2 - a_1$, $\omega = (\omega_1, \omega_2)$. If the slope of the line segment A_1A_2 is positive and less than* 1, *then there exist ω-homogenous forms H, G and a positive integer d such that $f_\omega^+ = \ominus H^d$ and $J(H, G) = \ominus H^s$, $s = 0$ or* 1.

Proof. By Lemma 2.2, $a_1 = 0$, $b_1 = m$ so that $\omega_1 = m - b_2$, $\omega_2 = a_2$. By Lemma 1.4 there exists ω-homogeneous elements H, G of $k[x, y]$, a positive integer d and a nonnegative integer e such that $f_\omega^+ = \ominus H^d$ and $J(H, G) = \ominus H^e$. By Lemma 2.1, f_ω^+ has the form $f_\omega^+ = \ominus_1 y^m + \cdots + \ominus_2 x^{a_2}y^{b_2}$, $\ominus_1\ominus_2 \neq 0$. By hypothesis, $(b_2 - m)/a_2 > 0$. Hence $b_2 > m$. Then $f_\omega^+ = y^m(\Theta_1 + \cdots + \Theta_2 x^{a_2} y^{b_2 - m})$. Let $d = \gcd(a_2, b_2 - m)$. Let $a = a_2/d$, $b = (b_2 - m)/d$. Then $f_\omega^+ = y^m F(x^a y^b)$, for some $F \in k[t]$ and $F(0) \neq 0$. Thus $H = y^q F_0(x^a y^b)$ with $F_0 \in k[t]$, $F_0^d = F$ and $dq = m$. Also G is of the form $G = x^{p'} y^{q'} G_0(x^a y^b)$, with $G_0 \in k[t]$, p', $q' \in \mathbb{Z}^+$, $G_0(0) \neq 0$.

By hypothesis, $b_2 - m - a_2 < 0$, hence $b - a < 0$ and $-aq(b-a) > 0$. If $p' = 0$, then by Lemma 4.1, H^{e-1} divides G. Then if $e > 0$, we replace G by G/H^{e-1} and obtain $J(H, G) = \Theta H$. Therefore we must only show that $p' \neq 0$.

If $p' = 0$, we obtain by (6),

$$\Theta x y^{q(e-1) - q' + 1} F_0^e = x^a y^b [aq' F_0' G_0 - aq F_0 G_0'] . \tag{9}$$

Since $F_0(0) \neq 0$, (9) implies either $a = 0$ or 1, by comparing lowest-degree terms on both sides of the equality, $a = 0$ implies $F_0 = 0$, thus $a = 1$. Since $b < a$, $b = 0$. Then $b_2 - m = 0$, which implies $\overline{A_1 A_2}$ is horizontal. Contradiction. □

Remark 4.4. Proposition 4.3 suggests the study of polynomials H, $G \in k[x, y]$ of the form $H = y^q H_0(x^a y^b)$, $G = x^{p'} y^{q'} G_0(x^a y^b)$ such that H_0, $G_0 \in k[t]$, $H_0(0) G_0(0) \neq 0$ and $J(H, G) = \Theta H^s$, $s = 0$ or 1. By (7), $p' = q' = 1$.

The hope would be, but this seems too optimistic, that this would force H_0 to be an element of k. Then if $\overline{A_1 A_2}$ has positive slope, this slope would have to be greater than 1 and by symmetry, if $\overline{A_{s-1} A_s}$ has positive slope, then its slope would have to be less than 1.

We might ask if $J(H, G) = \Theta H^s$, $s = 0$ or 1, implies some restriction as to the number of prime factors of H_0 in $\bar{k}[t]$.

5. Remarks on characteristic $p \neq 0$

In [3] we proved the following:

Proposition 5.1. *Let f, $g \in \mathbb{C}[x, y]$. If $J(f, g) = \Theta$, then for all but a finite number of prime numbers $p \neq 0$, there exists a finite field k of characteristic p and a pair of elements $\tilde{f}$, $\tilde{g} \in k[x, y]$ such that $S(f) = S(\tilde{f})$, $S(g) = S(\tilde{g})$ and $J(\tilde{f}, \tilde{g}) = 1$.* □

Proposition 5.2. *Let f, g, A_k, $0 \leq k \leq s$, be as in Definition 2.0. If $J(f, g) = \Theta$, then for each $k = 1, \ldots, s$, $a_k \neq b_k$.*

Proof. Suppose $a_k = b_k$ for some k, $1 \leq k \leq s$. Then there is a prime number $p > a_k$, a field of characteristic p, and $\tilde{f}$, $\tilde{g} \in k[x, y]$, such that $S(f) = S(\tilde{f})$, $S(g) = S(\tilde{g})$ and $J(\tilde{f}, \tilde{g}) = 1$. Let $D = \tilde{f}_y(\partial/\partial x) - \tilde{f}_x(\partial/\partial y)$. Then $D(\tilde{g}) = 1$ implies $D^p = 0$, which yields $\nabla(\tilde{f}^s) = 0$ for all $s \in \mathbb{Z}^+$ by Theorem 1.10, where $\nabla = \partial^{2p-2}/\partial x^{p-1} \partial y^{p-1}$. Let $\omega_1 = b_k - b_{k+1}$, $\omega_2 = a_{k+1} - a_k$, $\omega = (\omega_1, \omega_2)$. Then $\nabla(\tilde{f}_\omega^+)^s = 0$, for all $s \in \mathbb{Z}^+$. $\tilde{f}^+_\omega$ has the form $\Theta_1 x^{a_k} y^{b_k} + \cdots + \Theta_2 x^{a_{k+1}} y^{b_{k+1}} = f^+_\omega$ by Lemma 2.1. Since $p > a_k$, there exists r, $t \in \mathbb{Z}^+$ such that $ra_k = (p-1) + tp$. Then

$$\begin{aligned} 0 = \nabla(\tilde{f}^+_\omega)^r &= \Theta_1^r x^{ra_k} y^{rb_k} + \cdots + \Theta_2^r x^{ra_{k+1}} y^{rb_{k+1}} \\ &= \Theta_1^r x^{tp} y^{tp} + \cdots + \Theta_2^r \nabla(x^{ra_{k+1}} y^{rb_{k+1}}). \end{aligned}$$

Contradiction. □

Also in [3], we conjectured the following

5.3. Let k be a field of characteristic $p \neq 0$ and $f, g \in k[x, y]$ with $J(f, g) = \Theta$. Let $\omega = (\omega_1, \omega_2)$ be a gradation of x, y with $\omega_1 > 0$, $\omega_2 > 0$. If $\deg_\omega f = 0 \pmod p$, then $f^+_\omega(x^{\omega_1}, y^{\omega_2}) \in k[x^p, y^p]$.

I recently discovered the following counter-example when $p > 2$: Let $f = x^{(np-1)/2} y^{(mp+1)/2} + y$, $g = x^{(np+1)/2} y^{(mp-1)/2} - x$, where n, m are odd positive integers. With respect to the (1,1)-gradation, $f^+_\omega(x, y) \not\in k[x^p, y^p]$.

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